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THE STRUCTURE OF THE CONTROLLABILITY SET OF A LINEAR SUBCRITICAL SYSTEM

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Introduction

In the present paper we continue the research started in [1–3] and investigate the structure of the controllability set of the linear nonstationary system

$$\dot{x} = A(t)x + b(t)u, \qquad x \in \mathbf{R}^n, \qquad |u| \leq 1, \tag{1}$$

provided that the dual system $\dot{\psi} = -\psi A(t)$ has the nonoscillation property with respect to the hyperplane determined by the normal vector $b(t)$. Such systems are said to be subcritical [3]. This notion was introduced in the 50s by Azbelev in papers dealing with boundary value problems and differential inequalities: the maximal solvability interval of a fixed class of de la Vallée–Poussin problems was called subcritical in these papers. It turns out that the problem on the existence of a time-optimal positional control for system (1) is closely related to the solvability of some class of n-point problems (which, however, is not dealt with in the present paper); that is why we have borrowed the term.

In the following, we show that if a system is subcritical, then the boundary of its controllability set is a union of disjoint smooth manifolds [the smoothness is higher by 1 than that of the function $t \to (A(t), b(t))$] whose dimensions diminish from $n-1$ to 0; moreover, the union of manifolds whose dimensions grow from 0 to $k-1$ is the common boundary of the union of manifolds whose dimensions diminish from $n-1$ to k. The described structure of the boundary is naturally referred to as the Polya–Mammana factorization, since (as was mentioned above) the intrinsic causes of such a structure are related to the representation of an nth-order subcritical differential operator as the product of first-order differential operators.

Next, it turns out that the controllability set of a subcritical system also admits the Pola–Mammana factorization in the extended phase space [the space of (t, x)-variables]: it can be represented as a union of weakly invariant smooth manifolds whose dimensions diminish from $n+1$ to 1; moreover, the manifold of dimension k is the boundary of the closure of the manifold of dimension $k+1$.

Such a structure allows one to have a well-defined time-optimal positional control. Moreover, it suffices to specify the positional control only on the manifold of maximal dimension, since the motions $(t, x(t))$ (in the Filippov sense) of a system with right-hand side discontinuous in the phase coordinates are independent of the definition of the right-hand side on sets of zero Lebesgue measure in the extended phase space. Therefore, the existence of a Polya–Mammana factorization results in the existence of a time-optimal positional control.

1. Notation and Definitions

In the present paper we use the following notation: $\mathbf{R}^n$ is an n-dimensional Euclidean space with norm $|x| = \sqrt{x^*x}$ ($*$ stands for the transposition). Vector columns are denoted by Latin letters and vector rows are denoted by Greek letters unless otherwise specified (therefore, ξx stands for the inner product of the vectors ξ and x); $\operatorname{End}(\mathbf{R}^n)$ is the space of linear self-mappings of $\mathbf{R}^n$ equipped with the norm $|A| = \max\{|Ax| : |x| \leq 1\}$.

Let D be an arbitrary set in $\mathbf{R}^n$. We denote the interior of D with respect to $\mathbf{R}^n$ by $\operatorname{int} D$ and the closure of D in $\mathbf{R}^n$ by $\operatorname{cl} D$. The support function $\xi \to c(\xi, D)$ of the set D is given by the formula $c(\xi, D) =$

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$\sup\{\xi x : x \in D\}$. For the properties of the support function, see [4]. For our investigation it is important that the inclusion $0 \in \operatorname{int} D$ is equivalent to the validity of the inequality $c(\xi, D) > 0$ for all $\xi \in S^{n-1} \doteq \{\xi \in \mathbf{R}^n : \ |\xi| = 1\}$.

Recall [5] that a mapping f of M into N, where M and N are C^r-manifolds, $r \geq 1$, embedded in finite-dimensional spaces belongs to the class C^k, $k \leq r$, at a point $p \in M$ if for any k times continuously differentiable curve $p : (-1, 1) \to M$ passing through the point p $[p(0) = p]$, the function $\varepsilon \to f(p(\varepsilon))$, mapping $(-1, 1)$ into N, belongs to the class C^k at the point $\varepsilon = 0$. Next, the mapping $df(p) : T_pM \to T_qN$, where T_pM is the space tangent to M at the point p and T_qN is the space tangent to N at the point $q = f(p)$, defined by $df(p)v = df(p(\varepsilon))/d\varepsilon|_{\varepsilon=0}$ for any C^k-curve $p : (-1, 1) \to M$ $[p(0) = p,\ dp(\varepsilon)/d\varepsilon|_{\varepsilon=0} = v \in T_pM]$ is referred to as the derivative of the mapping f at the point p. A mapping $f : M \to N$ is called a C^k-diffeomorphism if it belongs to the class C^k and its inverse belongs to the same class. If $f : M \to N$ is a mapping of the class C^k and $df(p)$ is an isomorphism for each $p \in M$, then f is a diffeomorphism of the class C^k.

Everywhere in the following we assume that the functions $A : \mathbf{R} \to \operatorname{End}(\mathbf{R}^n)$ and $b : \mathbf{R} \to \mathbf{R}^n$ determining system (1) are continuous.

The optimal time function $(t, x) \to \tau_n(t, x)$ of system (1) is defined as the function whose value at each point (t_0, x_0) is given by the relation $\tau_n(t_0, x_0) = \min_{u(\cdot)\in\mathcal{U}}\{\vartheta \geq 0 : \ x(t_0 + \vartheta, t_0, x_0, u(\cdot)) = 0\}$, where $\mathcal{U}$ is the set of measurable functions with range $[-1, 1]$, and $x(t, t_0, x_0, u(\cdot))$ is the solution of system (1) with the control $u = u(t)$ and the initial condition $x(t_0) = x_0$. If for some point (t_0, x_0) there is no admissible control bringing the solution to zero in finite time, then we set $\tau_n(t_0, x_0) = \infty$.

The controllability set of system (1) on the closed interval $[t_0, t_0 + \vartheta]$ is defined as

$$D_\vartheta(t_0) = \{x \in \mathbf{R}^n : \ \tau_n(t_0, x) \leq \vartheta\}.$$

For $D_\vartheta(t_0)$ we have the relation (e.g., see [6, p. 103])

$$D_\vartheta(t_0) =: -\int_{t_0}^{t_0+\vartheta} X(t_0, t)\, b(t) U\, dt, \tag{2}$$

where $U = [-1, 1]$, $X(t, s)$ is the Cauchy matrix of the system $\dot{x} = A(t)x$ and the integral is treated in the Lyapunov sense [7, p. 229].

System (1) is said to be differentially controllable at a point t_0 if $0 \in \operatorname{int} D_\vartheta(t_0)$ for all $\vartheta > 0$ and differentially controllable on an interval $J \subset \mathbf{R}$ if it is so at each point of this interval. It was shown in [6, Lemma 1 and Theorem 2] that if system (1) is differentially controllable on J, then for any $t_0 \in J$ the controllability set $D(t_0) \doteq \bigcup_{\vartheta\geq 0} D_\vartheta(t_0)$ is open in $\mathbf{R}^n$ and the optimal time function is continuous at each point $(t_0, x_0) \in J \times D(t_0)$.

A set $\mathfrak{N}$ in the extended phase space $\mathbf{R}^{1+n}$ of system (1) is said to be weakly invariant if for any point $(t_0, x_0) \in \mathfrak{N}$ there exists a control $u_0 \in \mathcal{U}$ such that the solution $x_0(t)$ of system (1) with $u = u_0(t)$ and the initial condition $x_0(t_0) = x_0$ satisfies the inclusion $(t, x_0(t)) \in \mathfrak{N}$ for all $t \geq t_0$. In particular, the extended controllability set $\mathfrak{D} \doteq \mathbf{R} \times D_{\sigma(t)}(t)$ is weakly invariant [for any function $\sigma(t)$ satisfying the inequality $\sigma(t) > 0$ for all $t \in \mathbf{R}$].

2. The Subcritical Property and Nonoscillation

Let $\psi_1(t), \ldots, \psi_n(t)$ be an arbitrary principal solution system of the dual equation

$$\dot{\psi} = -\psi A(t). \tag{3}$$

By $\sigma(t_0)$ we denote the least upper bound of $\sigma > 0$ such that the system of functions

$$\xi_1(t) \doteq \psi_1(t)b(t), \quad \ldots, \quad \xi_n(t) \doteq \psi_n(t)b(t) \tag{4}$$

is a Chebyshev system (a T-system) on the half-open interval $[t_0, t_0 + \sigma)$. This means that any nontrivial linear combination of the functions (4) has at most $n - 1$ geometrically (i.e., with no regard of multiplicities) distinct zeros on $[t_0, t_0 + \sigma)$. It follows from the definition of $\sigma(t_0)$ that for any $\vartheta \in [0, \sigma(t_0)]$ each nontrivial solution of system (3) intersects the hyperplane $\gamma(t) \doteq \{\psi \in \mathbf{R}^n : \ \psi b(t) = 0\}$ at most $n - 1$ times as t ranges over the interval $[t_0, t_0 + \vartheta)$. This property was termed in [1] the *nonoscillation* of system (3) on the interval

$[t_0, t_0+\vartheta)$ with respect to the hyperplane $\gamma(t)$. Simple examples show that the function $t \to \sigma(t)$ (either taking nonnegative finite values or equal to $+\infty$) can be discontinuous, but the inequality $\sigma(t_0 - 0) \le \sigma(t_0) \le \sigma(t_0 + 0)$ is always valid. Note that the set (4) is a T-system on $[t_0, t_0+\vartheta)$ if and only if the determinant $\det\left(\xi_i(t_j)\right)_{i,j=1}^n$ is nonzero [8, p. 51] for any set of points $t_1, \ldots, t_n$ such that $t_0 \le t_1 < t_2 < \cdots < t_n < t_0 + \vartheta$.

The main property of T-systems required for our further considerations is known as the Bernstein theorem [8, p. 53]: *if the set (4) is a T-system on $[t_0, t_0+\vartheta)$, then for any set of points $t_1, \ldots, t_{n-1}$ such that $t_0 \le t_1 < t_2 < \cdots < t_{n-1} < t_0 + \vartheta$ there exists a linear combination of functions (4) which has simple zeros at the points t_i and does not have any other zero on $[t_0, t_0+\vartheta)$.*

Definition. *System (1) is said to be subcritical on the interval J if $\sigma(t) > 0$ for all $t \in J$.*

Lemma. *If system (1) is subcritical on J, then it is differentially controllable for all $t \in J$.*

Indeed, for each $t_0 \in J$ and any $\vartheta \in (0, \sigma(t_0))$ the support function $\psi \to c(\psi, D_\vartheta(t_0))$ of the set $D_\vartheta(t_0)$ is given by the relation $\int_{t_0}^{t_0+\vartheta} |\xi(t)|dt$, where $\xi(t) = \psi(t)b(t)$ and $\psi(t)$ is the solution of system (3) with $\psi(t_0) = -\psi$. Therefore, $\min_\psi \{c(\psi, D_\vartheta(t_0)) :\ \psi \in S^{n-1}\} > 0$, which completes the proof of the theorem.

Theorem 1. *A system of the form (1) is subcritical if it can be reduced by a nondegenerate transformation $z(t) = L(t)x$ [where $L(t)$ is continuously differentiable and $\det L(t) \ne 0$, $t \in J$] to the canonical system*

$$\dot z = F(t)z + g(t)u. \tag{5}$$

Here $F(t) = \{f_{ij}(t)\}_{i,j=1}^n$, $f_{ij}(t) = 0$ for $i > j+1$, is an upper triangular matrix with nonzero secondary diagonal consisting of the entries $f_{i+1,i}(t) = -\beta_{i+1}(t)$, $i = 1, \ldots, n-1$, $g(t) = \operatorname{colon}\big(\beta_1(t), 0, \ldots, 0\big) \in \mathbf{R}$; moreover, $f_{ik}(t)$ and $\beta_i(t)$ are continuous functions, and $\beta_i(t) > 0$ for $t \in J$ and $i = 1, \ldots, n$.

Proof. By $\sigma(t; A, b)$ we denote the function $\sigma(t)$ constructed on the basis of system (1). Let us show that $\sigma(t; A, b)$ is invariant under the nondegenerate transformation $z = L(t)x$ [i.e., $\sigma(t; A, b) = \sigma(t; F, g)$, where $F = \left(\dot L + LA\right)L^{-1}$ and $g = Lb$]. Indeed, the solution of system (3) and that of the system

$$\dot\eta = -\eta F(t), \tag{6}$$

dual to the system $\dot z = F(t)z$, are related by the formula $\psi(t) = \eta(t)L(t)$; therefore [see (4)], $\xi_i(t) = \psi_i(t)b(t) = \eta_i(t)L(t)b(t) = \eta_i(t)g(t)$.

Suppose that F and g are taken from system (5); let us show that $\sigma(t; F, g) > 0$ for all $t \in J$. Note that, first, the condition $\beta_1(t) > 0$ yields the relation $\sigma(t; F, g) = \sigma(t; F, e_1)$, where $e_1 = \operatorname{colon}(1, 0, \ldots, 0)$; therefore, in the following we assume that $g = e_1$. Second, we can assume that all functions occurring on the main diagonal of the matrix F vanish. Indeed, the nondegenerate transformation $\xi^{(i)} = \eta^{(i)} \exp \int_0^t f_{ii}(s)ds$, $i = 1, \ldots, n$ (the bracketed superscript stands for the corresponding coordinate of the vector), reduces system (6) to the form $\dot\xi = -\xi F^0(t)$, where the main diagonal of the matrix F^0 consists of zero entries.

For each $k = 3, \ldots, n$ we introduce the matrices $U_k(t) = (m(t), e_2, \ldots, e_k)^* \in \operatorname{End}(\mathbf{R}^n)$, where $m(t) = \operatorname{colon}\left(\mu^{(1)}(t), \ldots, \mu^{(k)}(t)\right)$, and $F_{k-1}(t) = \{q_{ij}(t)\}_{i,j=1}^{k-1} \in \operatorname{End}\left(\mathbf{R}^{k-1}\right)$, where $q_{ij}(t) = 0$ for $i > j+1$ and $i = j$, whereas $q_{i+1,i}(t) = -\beta_{n-k+i+2}(t)$.

Let $\mu(t)$ be the solution of system (6) with the initial condition $\mu(t_0) = e_1$, $t_0 \in J$, and let ε_1 be a number such that $\mu^{(1)}(t) > 0$ for all $t \in J_1 \doteq [t_0, t_0 + \varepsilon_1]$. On J_1 we perform the nondegenerate transformation $\eta = \xi U_n(t)$; then $\eta^{(1)} = \mu^{(1)}(t)\xi^{(1)}$, $\eta^{(i)} = \mu^{(i)}(t)\xi^{(1)} + \xi^{(i)}$, $i = 2, \ldots, n$, and system (6) for ξ acquires the form

$$\dot\xi^{(1)} = \xi^{(2)}\beta_2(t)/\mu^{(1)}(t), \qquad \dot\eta_1 = -\eta_1 F_{n-1}(t), \qquad \eta_1 = \left(\xi^{(2)}, \ldots, \xi^{(n)}\right) \in \mathbf{R}^{n-1} \tag{7}$$

[the transformation $\eta = \xi U_n(t)$ does not guarantee that the matrix $F_{n-1}(t)$ has the zero diagonal, but, as was mentioned above, this diagonal can be reduced to zero].

The performed transformation has the following property. If system (6) has a nontrivial solution $\eta(t)$ such that the first coordinate $\eta^{(1)}(t)$ vanishes at least n times on some interval $[t_0, t_0+\delta)$, $\delta \le \varepsilon$, then, by virtue of (7), there exists a nontrivial solution $\eta_1(t)$ of the system $\dot{\eta}_1 = -\eta_1 F_{n-1}(t)$ such that the first coordinate of this solution has at least $n-1$ zeros on $[t_0, t_0+\delta)$.

Continuing the reduction to the canonical form, on the basis of the system $\dot{\eta}_1 = -\eta_1 F_{n-1}(t)$ and the solution $\mu_1(t)$ of this system satisfying the condition $\mu_1(t_0) = (1, 0, \ldots, 0) \in \mathbf{R}^{n-1}$ we construct the new system $\dot{\eta}_2 = -\eta_2 F_{n-2}(t)$, $\eta_2 \in \mathbf{R}^{n-2}$, on the closed interval $J_2 \doteq [t_0, t_0+\varepsilon_2]$ on which the inequality $\mu_1^{(1)}(t) > 0$ is valid. This system has the following property: if system (6) has a solution whose first coordinate has at most n zeros on $[t_0, t_0+\delta)$, $\delta \le \min\{\varepsilon_1, \varepsilon_2\}$, then the system $\dot{\eta}_2 = -\eta_2 F_{n-2}(t)$ has a solution whose first coordinate vanishes at least $n-2$ times on $[t_0, t_0+\delta)$.

At the last step, the canonical system becomes the equation $\dot{\eta} = 0$, whose arbitrary nontrivial solution has no zeros. The proof of the theorem is complete.

Suppose that system (1) satisfies the following two conditions.

Condition 1. For each $i = 1, \ldots, n+1$ the functions $t \to q_i(t)$ given by the formulas $q_1(t) = b(t)$, ..., $q_i(t) = \dot{q}_{i-1}(t) - A(t)q_{i-1}(t)$ are continuous and bounded on $\mathbf{R}$ and satisfy the relation $\det Q(t) \neq 0$ for all $t \in \mathbf{R}$, where $Q(t) = (q_1(t), \ldots, q_n(t))$.

Condition 2. There exist numbers $\nu_1, \ldots, \nu_{n-1}$ such that $\nu_1 \le \nu_2 \le \cdots \le \nu_{n-1}$ and the roots $\lambda_1(t), \ldots, \lambda_n(t)$ of the equation $\det(\lambda Q(t) - H(t)) = 0$, where $H(t) = (q_2(t), \ldots, q_{n+1}(t))$, satisfy the inequalities

$$\lambda_1(t) \le \nu_1 \le \lambda_2(t) \le \cdots \le \nu_{n-1} \le \lambda_n(t) \tag{8}$$

for all t.

Theorem 2. *If Conditions 1 and 2 are satisfied, then $\sigma(t) = \infty$ for all $t \in \mathbf{R}$. Next, if there exist constants $\varepsilon > 0$ and $\delta \ge 0$ such that, in addition to (8), $\delta \le \lambda_1(t)$, $\nu_{i-1} + \varepsilon \le \lambda_i(t) \le \nu_i - \varepsilon$, $i = 2, \ldots, n-1$, for all sufficiently large t, then the controllability set $D(t)$ of system (1) coincides with $\mathbf{R}^n$ for all $t \in \mathbf{R}$.*

Proof. Let $r(t) = \operatorname{colon}(r_1(t), \ldots, r_n(t))$ be a solution of the algebraic system $Q(t)r = q_{n+1}(t)$. Straightforward verification shows that the substitution $z = L(t)x$, $L = Q^{-1}$, reduces system (1) to system (5), where $\beta_i(t) \equiv 1$, $f_{in}(t) = -r_i(t)$, $i = 1, \ldots, n$, and $f_{ik}(t) \equiv 0$, $i \le k$, $i, k = 1, \ldots, n-1$. Hence $\sigma(t) > 0$. Further, we can readily see that system (6) is equivalent to the equation

$$\xi^{(n)} = r_1(t)\xi + \cdots + r_n(t)\xi^{(n-1)}. \tag{9}$$

Let us show that $\sigma(t) = \infty$. By virtue of Corollary 5.3 in [9], if there exist numbers $\nu_0, \ldots, \nu_{n-1}$ such that $\nu_0 < \nu_1 < \cdots < \nu_{n-1}$ and the roots $\mu_1(t), \ldots, \mu_n(t)$ of the characteristic equation

$$\mu^n = r_1(t) + r_2(t)\mu + \cdots + r_n(t)\mu^{n-1} \tag{10}$$

satisfy inequalities (8) (with μ_i replaced by λ_i) and the inequality $\nu_0 \le \mu_1(t)$, then for Eq. (9) we have $\sigma(t) = \infty$ for all $t \in [0, +\infty)$. Moreover, there exists a principal solution system of Eq. (9) such that $c_i \exp(\nu_{i-1}t) \le \xi_i(t) \le d_i \exp(\nu_i t)$, $i = 1, \ldots, n-1$, and $c_n \exp(\nu_{n-1}t) \le \xi_n(t)$, $t \in [0, \infty)$, where c_i and d_i are some positive constants.

Let us show that $\mu_i(t) = \lambda_i(t)$. Indeed, the roots of Eq. (10) are the eigenvalues of the matrix $\hat{H}(t) \doteq (e_2, \ldots, e_n, r(t))$, where e_i is the ith unit vector; therefore, $\det\left(\mu(t)I - \hat{H}(t)\right) = 0$ for each root $\mu(t)$ of Eq. (10). Consequently, $\det Q(t)\left(\mu(t)I - \hat{H}(t)\right) = \det\left(\mu(t)Q(t) - Q(t)\hat{H}(t)\right)$. Since $Q(t)\hat{H}(t) = H(t)$, we have $\det(\mu(t)Q(t) - H(t)) = 0$. Therefore, any root of Eq. (10) is simultaneously a solution of the equation $\det(\lambda Q(t) - H(t)) = 0$, whence $\sigma(t) = \infty$.

Let us show that $D(t) = \mathbf{R}^n$. Since the function $\xi(t) = -\psi X(t_0, t)\, b(t)$ is a solution of Eq. (9), it follows that the support function $c(\psi, D_\vartheta(t))$ of the set $D_\vartheta(t)$ has the form $c(\psi, D_\vartheta(t)) = \int_t^{t+\vartheta} |\xi(s)|ds$. By virtue of the inequality $\delta \le \lambda_1(t)$, occurring in the assumption of Theorem 2, and the results of [9, Corollary 5.3], there exists an $\alpha > 0$ such that $|\xi(t)| \ge \alpha$; therefore, $c(\psi, D_\vartheta(t)) \to \infty$ as $\vartheta \to \infty$ for any $\psi \neq 0$. The proof of the theorem is complete.

Example 1. Let us consider the system of equations $\dot{x}_1 = a_1(t)x_1 + a_2(t)x_3 + u$, $\dot{x}_2 = x_1$, $\dot{x}_3 = x_1 + a_3(t)x_3$ describing (in the linear approximation) the dynamics of an aircraft with variable aerodynamic characteristics [10]. Here x_2 is the pitch angle, x_3 is the angle of attack, and u is the elevator application. We can directly justify that if the a_i are independent of t, then the inequality $a_3 \neq 0$ is a necessary and sufficient condition for the system to be subcritical, and the conditions $a_3 \neq 0$ and $(a_1 - a_3)^2 + 4a_2 \geq 0$ provide the global subcritical property (i.e., the validity of the relation $\sigma = \infty$). Now let $t \to a_i(t)$ be continuous functions, and let $a_3(t) \neq 0$ for all t. In this case, using Theorems 1 and 2, we can show that if for each t there exists a $\vartheta > 0$ such that

$$-1 \leq a_3(t) \int\limits_t^{t+\vartheta} a_2(s) \exp \int\limits_t^s (a_3(\tau) - a_1(\tau))\, d\tau\, ds \leq 0,$$

then the system is subcritical; moreover, $\sigma(t) > \vartheta$. In particular, if either $a_3(t) < 0$ and $a_2(t) \geq 0$ or $a_3(t) > 0$ and $a_2(t) \leq 0$, then $\sigma(t) > 0$.

3. The Structure of the Boundary of the Controllability Set

In the following we assume that the functions $A : \mathbf{R} \to \operatorname{End}(\mathbf{R}^n)$ and $b : \mathbf{R} \to \mathbf{R}$ determining system (1) are bounded on $\mathbf{R}$ and belong to the class C^r (i.e., are r times continuously differentiable on $\mathbf{R}$), where $r \geq 0$, and system (1) is subcritical.

If $\vartheta \leq \sigma(t_0)$, then, by virtue of the Pontryagin maximum principle

$$\max_{u(\cdot)\in\mathcal{U}} \psi(t)b(t)u = \psi(t)b(t)u(t), \qquad t_0 \leq t \leq t_0 + \vartheta, \tag{11}$$

for any point $x_0 \in D_\vartheta(t_0)$ there exists an integer k, $0 \leq k \leq n-1$, and a vector $\tau \in M^k(\vartheta)$, where $M^0(\vartheta) \doteq \{0\}$ and $M^k(\vartheta) \doteq \{\tau = (\tau_{n-k}, \ldots, \tau_{n-1}) \in \mathbf{R}^k : \ 0 < \tau_{n-k} < \cdots < \tau_{n-1} < \vartheta\}$, $k = 1, \ldots, n-1$, such that the control bringing the solution $x_0 = x(t_0)$ to the origin in minimum time takes the values $+1$ and -1 and has switchings only at the points $t_0 + \tau_i$, $i = n-k, \ldots, n-1$ [the points $t_0 + \tau_i$, $i = n-k, \ldots, n-1$, correspond to the zeros of the function $\xi(t) \doteq \psi(t)b(t)$, where $\psi(t)$ is some nontrivial solution of system (3)]. By virtue of the condition $\vartheta < \sigma(t_0)$, there are at most $n-1$ switchings. The set $M^0(\vartheta)$ corresponds to the points from $D_\vartheta(t_0)$ that are brought to zero by controls without switchings. We treat $M^k(\vartheta)$ as embedded (in $\mathbf{R}^k$) smooth manifolds of dimension k equipped with the natural topology.

For each $k = 0, \ldots, n-1$ we construct the sets $N_+^k(t_0, \vartheta)$ and $N_-^k(t_0, \vartheta)$ as follows: $N_+^k(t_0, \vartheta)$ consists of all points $x_0 \in D_\vartheta(t_0)$ for each of which there exists a point $\tau(t_0, x_0) \in M^k(\vartheta)$ such that the optimal control $u(t, x_0)$, $t_0 \leq t \leq t_0 + \vartheta$, brings the system from the point $x(t_0) = x_0$ to the point $x(t_0 + \vartheta) = 0$ and has switchings only at the instants $t = t_0 + \tau_i(t_0, x_0)$, and moreover, $u(t, x_0) = +1$ before the first switching. In this case, the set $N_+^0(t_0, \vartheta)$ is a singleton: $N_+^0(t_0, \vartheta) = \left\{ -\int\limits_{t_0}^{t_0+\vartheta} X(t_0, t)\, b(t) dt \right\}$. The sets $N_-^k(t_0, \vartheta)$ are defined in a similar way with replacing the control $u(t, x_0)$ by -1 for $t_0 \leq t < t_0 + \tau_{n-k}(t_0, x_0)$. The sets $N_+^k(t_0, \vartheta)$ and $N_-^k(t_0, \vartheta)$, $k = 0, \ldots, n-1$, have the following properties.

Property 1. *Let $\vartheta \leq \sigma(t_0)$. Then $N_+^k(t_0, \vartheta) \subset \partial D_\vartheta(t_0)$, and any point $x_0 \in N_+^k(t_0, \vartheta)$ corresponds to a unique point $\tau(t_0, x_0) \in M^k(\vartheta)$ (that is, the set of switching points) such that the control $u(t, x_0)$ satisfying the maximum principle (11) brings the system from the point $x_0 = x(t_0)$ to the point $x(t_0 + \vartheta) = 0$.*

Proof. Indeed, since $N_+^k(t_0, \vartheta) \subset D_\vartheta(t_0)$, it follows that the existence of a point $x_0 \in N_+^k(t_0, \vartheta)$ that does not belong to $\partial D_\vartheta(t_0)$ results in the inclusion $x_0 \in \operatorname{int} D_\vartheta(t_0)$. Therefore, the optimal time $\tau_n(t_0, x_0)$ satisfies the inequality $\tau_n(t_0, x_0) < \vartheta$. The corresponding control $u_0(t)$ satisfies condition (11) for some nontrivial solution $\psi_0(t)$ of Eq. (3). On the other hand, it follows from the definition of $N_+^k(t_0, \vartheta)$ that there exists a $\tau \in M^k(\vartheta)$ such that the control $u(t, \tau)$ also brings x_0 to the origin in time ϑ. We complete the definition of $u_0(t)$ on $(t_0 + \tau_n(t_0, x_0), t_0 + \vartheta]$ by identical zero and consider the function $v(t) = u(t, \tau) - u_0(t)$. We can readily see that this function preserves the sign on the intervals $(t_0 + \tau_i, t_0 + \tau_{i+1})$, $i = n-k-1, \ldots, n-1$, $\tau_{n-k-1} = 0$, $\tau_n = \vartheta$; moreover, $(-1)^i v(t) \geq 0$ for $t \in (t_0 + \tau_i, t_0 + \tau_{i+1})$. Consequently, the function $v(t)$ changes its sign at most k times on $[t_0, t_0 + \vartheta]$, i.e., there exist at most k distinct points $t_1 < t_2 < \cdots < t_k$ of the interval $(t_0, t_0 + \vartheta)$ such that $v(t_i)\, v(t_{i+1}) < 0$. Next, the function $y(t) \doteq x(t) - x_0(t)$ [where $x(\cdot)$ and $x_0(\cdot)$ are solutions of system (1) issuing from the point $x_0 = x(t_0)$ and corresponding to the controls $u(\cdot)$ and $u_0(\cdot)$] is a solution of the problem $\dot{y} = A(t)y + b(t)v(t)$, $y(t_0) = y(t_0 + \vartheta) = 0$, whence

$$\int\limits_{t_0}^{t_0+\vartheta} X(t_0, t)\, b(t) v(t) dt = 0. \tag{12}$$

Next, by the Bernstein theorem, there exists a vector $\psi \neq 0$ such that the function $\xi(t) = \psi X(t_0, t)\, b(t)$ has zeros at the points where $v(t)$ changes its sign, and $\xi(t)v(t) \geq 0$ for $t_0 \leq t \leq t_0 + \vartheta < t_0 + \sigma(t_0)$. Multiplying (12) by ψ, we obtain $\int\limits_{t_0}^{t_0+\vartheta} \xi(t)v(t)dt = 0$. Consequently, $v(t) \equiv 0$, whence $u(t,\tau) \equiv u_0(t)$ for $t_0 \leq t \leq t_0 + \vartheta$, which contradicts the assumption $\tau_n(t_0, x_0) < \vartheta$.

The uniqueness of the control $u(t,\tau)$ corresponding to the point $x_0 \in N_+^k(t_0, \vartheta)$ can be proved in a similar way: if the controls $u(t,\tau)$ and $u_0(t)$ bring the system from $x(t_0) = x_0$ to $x(t_0 + \vartheta) = 0$, then we have relation (12), where $v(t) = u(t,\tau) - u_0(t)$. The proof of the property is complete.

By virtue of Property 1, for each $\vartheta \leq \sigma(t_0)$ and any fixed $k = 0, \ldots, n-1$ we have the function $f^{-1} : N_+^k(t_0, \vartheta) \to M^k(\vartheta)$, which takes each point $x \in N_+^k(t_0, \vartheta)$ to the point $\tau \in M^k(\vartheta)$ determining the switching points of the optimal control $u(t,\tau)$. The function $f^{-1} = f_k^{-1}$ depends on ϑ and the index k, but we do not emphasize this fact unless necessary.

Property 2. *The function f^{-1} is continuous and defines a homeomorphism of the sets $N_+^k(t_0, \vartheta)$ and $M^k(\vartheta)$. The inverse function $f : M^k(\vartheta) \to N_+^k(t_0, \vartheta)$ is given by the formula*

$$f(\tau) = \sum_{i=n-k-1}^{n-1} (-1)^{i-n+k} \int\limits_{t_0+\tau_i}^{t_0+\tau_{i+1}} X(t_0, t)\, b(t)dt, \tag{13}$$

where $\tau_{n-k-1} = 0$ and $\tau_n = \vartheta$.

Proof. Let us show that f^{-1} is a continuous function. Let $\{x_j\}_1^\infty \subset N_+^k(t_0, \vartheta)$ be a sequence such that $x_j \to x_0 \in N_+^k(t_0, \vartheta)$. The sequence $\{x_j\}_1^\infty$ corresponds to the sequence $\{\tau_j\}$, $\tau_j = f^{-1}(x_j) \in M^k(\vartheta)$, and the point x_0 corresponds to the point $\tau_0 = f^{-1}(x_0) \in M^k(\vartheta)$. We must show that $\tau_j \to \tau_0$. From the sequence $\{\tau_j\}_1^\infty$ we extract a convergent subsequence and denote it by $\{\tau_j\}_1^\infty$ again. The limit of this sequence is $\tau_* \in \operatorname{cl} M^k(\vartheta)$. Let the control $u(t, \tau_j)$ correspond to the point x_j (i.e., bring the system from x_j to the origin); then

$$x_j = -\int\limits_{t_0}^{t_0+\vartheta} X(t_0, t)\, b(t) u(t, \tau_j)\, dt. \tag{14}$$

The sequence $\{u(t, \tau_j)\}_1^\infty$ weakly converges to $u(t, \tau_*)$; therefore, by passing to the limit in (14), we obtain $x_0 = -\int\limits_{t_0}^{t_0+\vartheta} X(t_0, t)\, b(t) u(t, \tau_*)\, dt$. On the other hand, $x_0 = -\int\limits_{t_0}^{t_0+\vartheta} X(t_0, t)\, b(t) u(t, \tau_0)\, dt$, whence $\tau_* = \tau_0$ (see the proof of Property 1). This, together with the convergence $x_j \to x_0$, yields $f^{-1}(x_j) \to f^{-1}(x_0)$. The proof of (13) is obvious.

Property 3. *Let $\vartheta \leq \sigma(t_0)$. The vectors $h(\tau_{n-k}) \doteq X(t_0, t_0 + \tau_{n-k})\, b(t_0 + \tau_{n-k})$, $\ldots$, $h(\tau_{n-1}) \doteq X(t_0, t_0 + \tau_{n-1})\, b(t_0 + \tau_{n-1})$ are linearly independent for each $k = 1, \ldots, n-1$ and for any point $\tau = (\tau_{n-k}, \ldots, \tau_{n-1}) \in M^k(\vartheta)$.*

Proof. If there exist numbers $c_{n-k}, \ldots, c_{n-1}$ not simultaneously zero and such that $h(\tau_{n-k})\, c_{n-k} + \cdots + h(\tau_{n-1})\, c_{n-1} = 0$, then

$$\xi(\tau_{n-k})\, c_{n-k} + \cdots + \xi(\tau_{n-1})\, c_{n-1} = 0 \tag{15}$$

for any $\psi \in \mathbf{R}^n$, where $\xi(\tau_i) = \psi X(t_0, t_0 + \tau_i)\, b(t_0 + \tau_i)$. Let $c_j \neq 0$. There exists a vector $\psi \neq 0$ such that the function $\xi(t) = \psi X(t_0, t)\, b(t)$ has zeros at the points $t_0 + \tau_i$, $i \neq j$, and $\xi(\tau_j) \neq 0$ (this follows from the nonoscillation and the cited Bernstein theorem). This, together with (15), yields $\xi(\tau_j)\, c_j = 0$. The proof of the property is complete.

By virtue of Properties 1–3, for any $\vartheta \leq \sigma(t_0)$ and for each $k = 1, \ldots, n-1$ the set $N_+^k(t_0, \vartheta)$ is a smooth (of the class C^1) manifold of dimension k embedded in $\mathbf{R}^n$. In fact, it belongs to the class C^{r+1}. Let us prove this fact. The relation $f(\tau + \delta\tau) = f(\tau) + df(\tau)\delta\tau + o(|\delta\tau|)$, where $df(\tau) = (q_{n-k}(\tau), \ldots, q_{n-1}(\tau))$, $q_i(\tau) = \partial f(\tau)/\partial\tau_i = -2(-1)^{i-n+k} h(\tau_i)$, $i = n-k, \ldots, n-1$, means that the operator $df(\tau)$ acts from the space $T_\tau M^k$ tangent to $M^k(\vartheta)$ at the point τ (and identified with $\mathbf{R}^k$) into the space $T_x N^k$ tangent to $N_+^k(t_0, \vartheta)$ at the point $x = f(\tau)$ (and modeled by $\mathbf{R}^k$). In addition, $df(\tau)\left(T_\tau M^k\right) = T_x N^k$; therefore, $df(\tau)$ is an isomorphism. Taking into account the conditions imposed on system (1), we can readily see that the

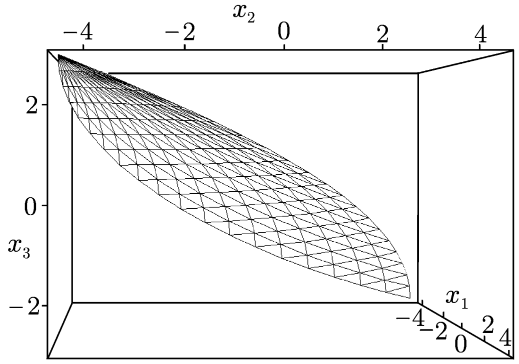

Fig. 1.

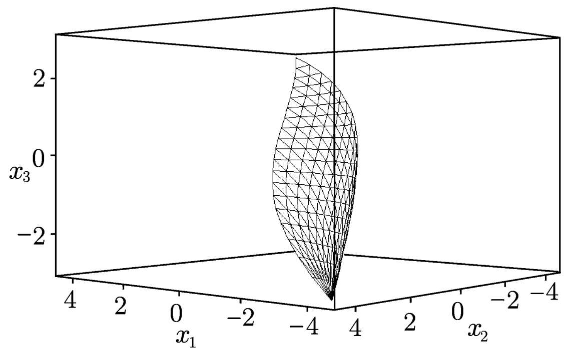

Fig. 2.

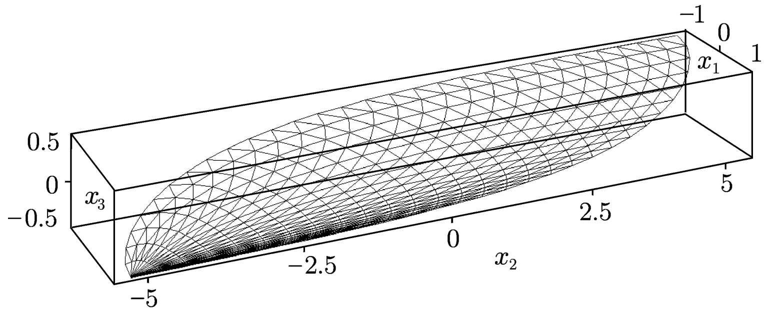

Fig. 3.

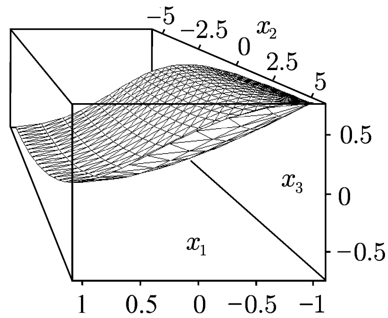

Fig. 4.

function $\tau \to df(\tau)$ belongs to the class C^r; consequently, f is a diffeomorphism of the class C^{r+1}; therefore, $N_+^k(t_0, \vartheta)$ is a manifold of the class C^{r+1}.

Theorem 3. *Let system (1) be subcritical on* $\mathbf{R}$. *Then for each* $\vartheta \le \sigma(t_0)$ *the controllability set* $D_\vartheta(t_0)$ *is a strictly convex body in* $\mathbf{R}^n$ *[i.e.,* $\operatorname{int} D_\vartheta(t_0) \ne \oslash$, *and* $\lambda x + (1-\lambda)x_0 \in \operatorname{int} D_\vartheta(t_0)$ *for any* $x, x_0 \in \partial D_\vartheta(t_0)$ *and any* $\lambda \in (0,1)$*]. The boundary* $\partial D_\vartheta(t_0)$ *of the set* $D_\vartheta(t_0)$ *is the union of disjoint smooth (of the class* C^{r+1}*) manifolds* $N_+^k(t_0, \vartheta)$ *and* $N_-^k(t_0, \vartheta)$, $k = 0, 1, \ldots, n-1$, *and the union* $\Big(\bigcup_{i=0}^{k-1} N_-^i(t_0, \vartheta)\Big) \cup \Big(\bigcup_{i=0}^{k-1} N_+^i(t_0, \vartheta)\Big)$ *is the common edge of the manifolds* $\operatorname{cl} N_+^k(t_0, \vartheta)$ *and* $\operatorname{cl} N_-^k(t_0, \vartheta)$. *Next, each point* $x \in N_+^k(t_0, \vartheta)$ *corresponds to a unique control bringing* $x(t_0) = x$ *to* $x(t_0 + \vartheta) = 0$; *moreover,* $u(t, x)$ *has exactly* k *switchings on the interval* $(t_0, t_0 + \vartheta)$.

Proof. All assertions of the theorem except for the strict convexity of $D_\vartheta(t_0)$ have already been proved. The strict convexity follows from the uniqueness of the control $u(t, x_0)$ bringing the point $x_0 = x(t_0) \in \partial D_\vartheta(t_0)$ to zero in time ϑ. Indeed, if there exists a $\lambda \in (0,1)$ such that $x_\lambda = \lambda x_1 + (1-\lambda)x_0 \in \partial D_\vartheta(t_0)$ for some $x_0, x_1 \in \partial D_\vartheta(t_0)$, then $u_\lambda(t) = \lambda u(t, x_1) + (1-\lambda)u(t, x_0)$ is the unique control bringing the point x_λ to the origin in time ϑ. Therefore, $|u_\lambda(t)| = 1$ for all $t \in [t_0, \vartheta)$. Consequently, either $\lambda \notin (0,1)$ or $u(t, x_0) = u(t, x_1)$, and therefore, $x_0 = x_1$.

Example 2. The controllability set $D_\vartheta(t_0)$ of the system $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = u$, $|u| \le 1$, is constructed in Fig. 1 for $\vartheta = 3$ and $t_0 = 0$. The "upper hat," that is, the manifold $N_+^2(0,3)$, is shown in Fig. 2.

Example 3. Figures 3 and 4 show the sets $D_\vartheta(t_0)$ and $N_+^2(t_0, \vartheta)$ for the system described in Example 1 in the case $|u| \le 1$, $t_0 = 0$, $\vartheta = 2\pi$, $a_1 = 1$, $a_2 = 0.1 \sin t$, and $a_3 = 1 + 0.999 \sin t$.

4. The Structure of the Extended Controllability Set

We write $\tau_n = \vartheta$, and for each $k = 0, 1, \ldots, n$ and any $t \in \mathbf{R}$ we introduce the manifolds $\mathcal{M}^k(t)$, where $\mathcal{M}^0(t) \doteq \{0\}$, $\mathcal{M}^k(t) \doteq \{\tau = (\tau_{n-k+1}, \ldots, \tau_n) : 0 < \tau_{n-k+1} < \cdots < \tau_n < \sigma(t)\}$, $k = 1, \ldots, n$, and $\mathcal{M}^{1+k} =$

$\mathbf{R} \times \mathcal{M}^k(t)$. To each point $p = (t, \tau) \in \mathcal{M}^{1+k}$ we assign the point $q = (t, x)$, where $x = 0$ for $k = 0$ and

$$x = x(p) = -\sum_{i=n-k}^{n-1} (-1)^{i-n+k} \int\limits_{t+\tau_i}^{t+\tau_{i+1}} X(t,s)b(s)ds, \qquad \tau_{n-k} = 0, \tag{16}$$

for $k \geq 1$.

Therefore, for each k we have the function $p \to F(p) = q$ with domain $\mathcal{M}^{1+k}$ and range $\mathcal{N}_+^{1+k} \doteq F\left(\mathcal{M}^{1+k}\right)$ (in the following we omit the subscript of $\mathcal{N}_+^1$). Since $\mathcal{N}_+^{1+k} = \mathbf{R} \times \mathcal{N}_+^k(t)$, where $\mathcal{N}^0(t) = \{0\}$, and for $k \geq 1$ the set $\mathcal{N}_+^k(t)$ consists of points of the form (16), we have $\mathcal{N}_+^k(t) \subset D_{\sigma(t)}(t)$. Therefore, by virtue of the Pontryagin maximum principle and the condition $\tau_n < \sigma(t)$, each point $q = (t, x) \in \mathcal{N}_+^{1+k}$ corresponds to a unique point $p = (t, \tau) \in \mathcal{M}^{1+k}$, which determines (for $k \geq 2$) the switchings of the time-optimal control bringing the position (t, x) to the position $(t + \tau_n, 0)$ (the optimal control identically vanishes for $k = 0$ and has no switchings for $k = 1$). This fact can be proved similarly to Property 1. Consequently, there exists a function $F^{-1} : \mathcal{N}_+^{1+k} \to \mathcal{M}^{1+k}$ inverse to F. Performing considerations similar to the proof of Property 2, we can readily find that $q \to F^{-1}(q)$ is a continuous function on $\mathcal{N}_+^{1+k}$. Therefore, F is a homeomorphism of the manifolds $\mathcal{M}^{1+k}$ and $\mathcal{N}_+^{1+k}$.

Next, F satisfies the relation $F(p + \delta p) = F(p) + dF(p)\delta p + o(|\delta p|)$, where $dF(p) = \operatorname{colon}(1, 0, \ldots, 0)$ for $k = 0$, and

$$dF(p) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ w_{n-k}(p) & w_{n-k+1}(p) & \cdots & w_n(p) \end{pmatrix} \tag{17}$$

for $k \geq 1$. Here

$$\begin{aligned} w_{n-k}(p) &= \partial x(p)/\partial t = A(t)x(p) + b(t) + w_{n-k+1}(p) + \cdots + w_n(p), \\ w_i(p) &= \partial x(p)/\partial \tau_i = 2(-1)^{i-n+k} X\left(t, t + \tau_i\right) b\left(t + \tau_i\right), \quad i = n - k + 1, \ldots, n - 1, \\ w_n(p) &= \partial x(p)/\partial \tau_n = (-1)^k X\left(t, t + \tau_n\right) b\left(t, t + \tau_n\right). \end{aligned}$$

We can show (just as in the proof of Property 3) that for each $p \in \mathcal{M}^{1+k}$ the vectors $w_{n-k+1}(p), \ldots, w_n(p)$ are linearly independent; therefore, so are the columns of the matrix $dF(p)$. Consequently, $dF(p)$ is an isomorphism of the space $T_p\mathcal{M}^{1+k}$ tangent to the manifold $\mathcal{M}^{1+k}$ at the point p onto the space $T_q\mathcal{N}_+^{1+k}$ tangent to the manifold $\mathcal{N}_+^{1+k}$ at the point $q = F(p)$. In addition, the function $p \to dF(p)$ belongs to the class C^r; therefore, F is a diffeomorphism of the class C^{r+1}. Consequently, for each $k = 0, 1, \ldots, n$ the manifold $\mathcal{N}_+^{1+k}$ is a smooth manifold of the class C^{r+1}.

The above-constructed manifolds $\mathcal{N}_+^{1+k}$ have the property that the time-optimal control $u_0(t)$ is equal to $+1$ for $t \in [t_0, t_0 + \tau_{n-k+1}(t_0, x_0))$, $(t_0, x_0) \in \mathcal{N}_+^{1+k}$. The manifolds $\mathcal{N}_-^{1+k}$, $k = 1, \ldots, n$, can be constructed in a similar way (in this case the optimal control starts from the value -1).

Remark. We can directly verify that, by virtue of (17), the velocity vector $v_+(q_0) \doteq \operatorname{colon}(1, A(t_0)x_0 + b(t_0))$ of the motion $t \to q(t) = (t, x(t))$ [where $x(t)$ is the solution of system (1) passing through the point x_0 at time t_0 under the control $u = +1$] lies in the space $T_{q_0}\mathcal{N}_+^{1+k}$ tangent to the manifold $\mathcal{N}_+^{1+k}$ at the point q_0. Next, it follows from the above constructions that the manifolds $\mathcal{N}^1$, $\mathcal{N}_-^{1+k}$, and $\mathcal{N}_+^{1+k}$, $k = 1, \ldots, n$, treated as manifolds embedded in $\mathbf{R}^{1+n}$, have no common points, and the manifold $\mathcal{N}_+^k \cup \mathcal{N}_-^k$ is the common boundary of the manifolds $\operatorname{cl} \mathcal{N}_+^{1+k}$ and $\operatorname{cl} \mathcal{N}_-^{1+k}$.

Theorem 4. *Let system (1) be subcritical. Then the extended controllability set* $\mathfrak{D} \doteq \mathbf{R} \times D_{\sigma(t)}(t)$ *can be represented in the form* $\mathfrak{D} = \operatorname{cl}\left(\mathfrak{N}_+^{1+n} \cup \mathfrak{N}_-^{1+n}\right)$, *where* $\mathfrak{N}_+^{1+k} = \mathcal{N}_+^{1+k} \cup \mathcal{N}_-^k \cup \mathcal{N}_+^{k-1} \cup \cdots \cup \mathcal{N}^1$ *and* $\mathfrak{N}_-^{1+k} = \mathcal{N}_-^{1+k} \cup \mathcal{N}_+^k \cup \mathcal{N}_-^{k-1} \cup \cdots \cup \mathcal{N}^1$, $k = 0, \ldots, n$. *The manifolds* $\mathfrak{N}_+^{1+k}$ *and* $\mathfrak{N}_-^{1+k}$ *are weakly invariant, and for each* $k = 0, \ldots, n$ *the manifold* $\mathfrak{N}_+^k \cup \mathfrak{N}_-^k$ *is the common boundary of the manifolds* $\operatorname{cl} \mathfrak{N}_+^{1+k}$ *and* $\operatorname{cl} \mathfrak{N}_-^{1+k}$.

Proof. To prove the weak invariance of the manifolds $\mathfrak{N}_+^{1+k}$, it suffices to note that for any point $q_0 = (t_0, x_0) \in \mathcal{N}_+^{1+k}$ there exists a control $u(t, q_0)$ (for example, one can always take the time-optimal control as u) such that the corresponding solution of system (1) with the initial condition $x(t_0) = x_0$ passes through the manifolds $\mathcal{N}_-^k$, $\mathcal{N}_+^{k-1}$, $\ldots$, $\mathcal{N}^1$ (see the remark). The weak invariance of the manifolds $\mathfrak{N}_-^{1+k}$ can be proved in a similar way. Next, as was shown above, the manifold $\mathcal{N}_+^k \cup \mathcal{N}_-^k$ is the common boundary of the manifolds

$\mathrm{cl}\,\mathcal{N}_+^{1+k}$ and $\mathrm{cl}\,\mathcal{N}_-^{1+k}$, $k = 1, \ldots, n$; consequently, for each $k = 0, \ldots, n$ the manifold $\mathfrak{N}_+^k \cup \mathfrak{N}_-^k$ is the common boundary of the manifolds $\mathrm{cl}\,\mathfrak{N}_+^{1+k}$ and $\mathrm{cl}\,\mathfrak{N}_-^{1+k}$.

Let us prove the representation $\mathfrak{D} = \mathrm{cl}\left(\mathfrak{N}_+^{1+n} \cup \mathfrak{N}_-^{1+n}\right)$. Let $q_0 \in \mathfrak{D}$. Then, by virtue of the definition of $\mathfrak{D}$, there exists a time-optimal control $u\,(t, q_0)$ bringing the motion $q\,(t, q_0) = (t, x\,(t, q_0))$ to the manifold $\mathcal{N}^1$ in time $\vartheta \leq \sigma\,(t_0)$; moreover, the solution $x\,(t, q_0)$ of system (1) has at most $n-1$ switchings. Consequently, $q\,(t, q_0)$ belongs to either $\mathrm{cl}\,\mathfrak{N}_+^{1+n}$ or $\mathrm{cl}\,\mathfrak{N}_-^{1+n}$ depending on the position of the point q_0, and we have the inclusion $\mathfrak{D} \subset \mathrm{cl}\left(\mathfrak{N}_+^{1+n} \cup \mathfrak{N}_-^{1+n}\right)$. Conversely, let $q_0 \in \mathrm{cl}\,\mathcal{N}_+^{1+n}$. Then there exists a time-optimal control $u\,(t, q_0)$ bringing $q\,(t, q_0)$ to the manifold $\mathcal{N}^1$ in time $\vartheta \leq \sigma\,(t_0)$. Performing similar considerations for $q_0 \in \mathrm{cl}\,\mathcal{N}_-^{1+n}$, we obtain the inclusion $\mathrm{cl}\left(\mathfrak{N}_+^{1+n} \cup \mathfrak{N}_-^{1+n}\right) \subset \mathfrak{D}$, and finally, $\mathfrak{D} = \mathrm{cl}\left(\mathfrak{N}_+^{1+n} \cup \mathfrak{N}_-^{1+n}\right)$. The proof of the theorem is complete.

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